HYPOELLIPTIC ORDINARY DIFFERENTIAL OPERATORS

BY

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ABSTRACT

Necessary and sufficient conditions for the hypoellipticity of an ordinary differential operator with C^∞ coefficients in a neighborhood of a zero of finite order of the leading term are given. A sufficient condition for such an operator to be in a certain Hörmander class is also given.

1. Introduction and statement of results

Let L be a differential operator with C^∞ coefficients. L is said to be hypoelliptic in an open set Ω if for every $u \in \mathscr{D}'(\Omega)$, $Lu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. Ordinary differential operators, $Lu = \sum_{j=0}^{m} a_j(x)u^{(j)}(x)$, are hypoelliptic in every open set where $a_m(x)$ is different from zero, i.e., at all regular points of L. The problem of determining whether a given ordinary differential operator L is hypoelliptic in a neighborhood of a zero of $a_m(x)$ has not been considered explicitly in the literature. It is the aim of the present paper to characterize the class of all hypoelliptic ordinary differential operators for which $a_m(x)$ does not have a zero of infinite order (this class contains all hypoelliptic operators with analytic coefficients). As an illustration of this characterization, we shall give an example of a first order hypoelliptic operator, which is contained in no previously known class of hypoelliptic operators.

In order to state our main results, let us recall briefly some notions and facts from the classical theory of singular points of ordinary differential operators (a more detailed description may be found in Section 2). Let $a_j(x)$, $0 \leqq j \leqq m$, be C^∞ functions in a neighborhood of a certain zero of $a_m(x)$ (a possibly singular point of L), which we take to be the origin, and we assume that $a_m(x)$ vanishes at most to a finite order at $x = 0$. It is well known ([7], [8]) that there exists a system of m linearly independent formal solutions $u_1(x), \cdots, u_m(x)$ of $Lu = 0$ with

$$u_i(x) = e^{Q_i(x)} x^{\rho_i} v_i(x) \tag{1.1}$$

where the $Q_i(x)$ are polynomials in x^{-1/q_i} and

$$v_i(x) = \sum_{j=0}^{m_i} v_{i,j}(x)(\log x)^j \tag{1.2}$$

$$v_{i,j}(x) \sim \sum_{n=0}^{\infty} v_{i,j,n} x^{n/q_i} \tag{1.3}$$

for $0 \leqq j \leqq m_i$, $1 \leqq i \leqq m$. Here m_i, q_i are integers ($(m_i \geqq 0,\ q_i > 0)$ and the series in (1.3) do not converge, in general, even if the $a_j(x)$ are analytic. (The equations $Lu_i = 0$ hold in the sense of formal power series; the coefficients $a_j(x)$ are replaced by their formal Taylor expansions at the origin.) The functions $Q_i(x)$ are called "determining factors". Clearly, we can assume that the constant terms vanish in the determining factors. Let r denote the number of the determining factors which vanish identically, and assume that $Q_1(x) \equiv \cdots \equiv Q_r(x) \equiv 0$. We shall prove

THEOREM 1. *L is hypoelliptic in a neighborhood of the origin if and only if* (i) $a_r(0) \neq 0$ *and* (ii) $|\operatorname{Re} Q_i(x)| \to \infty$ *as* $x \to 0$ *for* $r < i \leqq m$.

While it is true that the determining factors are computable from the coefficients $a_j(x)$ (see [7], [8]), the actual verification of condition (ii) of Theorem 1 is not easy, in the most general case. In many cases, however, the conditions of Theorem 1 can be given a more explicit form.

Thus, in accordance with the usual notation in the literature on partial differential equations, set $p(x,\xi) = \sum_{j=0}^{m} a_j(x)(i\xi)^j$ — the (full) symbol of the operator L. It is well known that there exist m complex valued functions $\zeta_1(x), \cdots, \zeta_m(x)$, which are continuous in a neighborhood of the origin except at $x = 0$, such that $p(x,\xi) = a_m(x)\prod_{j=1}^{m}(\xi - \zeta_j(x))$. The functions $\zeta_1(x), \cdots, \zeta_m(x)$ are called branches of roots of p. We shall say that the operator L satisfies the condition (S) if, whenever $\zeta_i(x)$ is an unbounded branch of roots of $p(x,\xi)$ and $\zeta_i(x)/\zeta_j(x) \to 1$ as $x \to 0$, then $i = j$ (i.e., all unbounded branches are "asymptotically simple".) If condition (S) is satisfied, then the determination of the determining factors is relatively easy, and we shall prove

THEOREM 2. *Let L satisfy the condition (S). A necessary and sufficient condition for the hypoellipticity of L at a neighborhood U of the origin is that there exists a constant* $C > 0$ *such that for x in U and* ζ *a complex number satisfying* $p(x,\zeta) = 0$, *either* $|\zeta| < C$ *or* $|x||\operatorname{Im}\zeta| \to \infty$ *as* $x \to 0$.

Another class of hypoelliptic operators is exhibited in the following theorem.

THEOREM 3. *If there exist a neighborhood U of the origin and a constant* $C>0$ *such that for x in U,* ζ *a complex number satisfying* $p(x,\zeta)=0$, *either* (i) $|\zeta| < C$ *or* (ii) $|x|\,|\operatorname{Im}\zeta| \to \infty$ *and* $C|\operatorname{Im}\zeta| > |\operatorname{Re}\zeta|$ *as* $x \to 0$, *then there exist constants* $0 \leqq \delta < \rho \leqq 1$ *and a real number m', such that for every pair* (α,β) *of non-negative integers and for every compact subset K of U, there exist constants* $C_1(K)$ *and* $C(\alpha,\beta,K)$ *such that the inequalities*

(a) $$|p(x,\xi)| \geqq C_1(K)|\xi|^{m'}$$

(b) $$\left|\frac{\partial^{\alpha+\beta}p(x,\xi)}{\partial\xi^\alpha\partial x^\beta}\right| \leqq C(\alpha,\beta,K)(1+|\xi|)^{-\rho\alpha+\delta\beta}|p(x,\xi)|$$

hold for $x \in K$, ξ *a real vector,* $|\xi| \geqq C_1(K)$.

It follows from theorem 4.2 in [5] that if (a) and (b) are satisfied, then L is hypoelliptic. The class of operators described in Theorems 1 and 2 is strictly bigger than the class of operators described in Theorem 3. In fact, consider the first order operator $Lu = x^3u' + (i+x)u$. Then $p(x,\xi) = ix^3\xi + i + x$, $\partial p/\partial x = 3ix^2\xi + 1$, $\partial p/\partial\xi = ix^3$. For $\xi = -x^{-3}$ we get $p = -\xi^{-\frac{1}{3}}$, $\partial p/\partial x = -3i\xi^{\frac{2}{3}} + 1$, $\partial p/\partial\xi = -i\xi^{-1}$. If condition (b) were true for a compact neighborhood K of the origin, we would have $|\partial p/\partial x||\partial p/\partial\xi||p|^{-2} \leqq C(1,0,K)C(0,1,K)(1+|\xi|)^{-\rho+\delta} \to 0$ as $\xi \to \infty$, since $\delta < \rho$. But for the specified pairs of (x,ξ), $|\partial p/\partial x||\partial p/\partial\xi||p|^{-2} \sim 3|\xi|^{\frac{2}{3}}|\xi|^{-1}|\xi|^{\frac{2}{3}} = 3$ as $\xi \to \infty$, a contradiction.

It follows, however, form Theorem 1, that L is hypoelliptic. We know of no previously investigated class of hypoelliptic operators which contains L.

As a particular case of Theorem 1, we note that an ordinary differential operator can be hypoelliptic in no neighborhood of a regular singular point. We note also that if $Lu = xMu$ where M is an operator with C^∞ coefficients, then L is not hypoelliptic in any neighborhood of the origin, since then $a_r(0) = 0$.

It is well known [3] that for every non-elliptic hypoelliptic differential operator L with constant coefficients there exists a hyperfunction u such that $Lu \in C^\infty(\Omega)$ but $u \notin C^\infty(\Omega)$ where Ω is an open set (so that u cannot be a distribution in Ω). All ordinary differential operators with constant coefficients are elliptic. We shall show that for many, though not for all, non-elliptic hypoelliptic ordinary differential operators with analytic coefficients, there exists a non-C^∞ hyperfunction solution of $Lu = 0$ with $\operatorname{supp} u \neq \{0\}$. (Thus, e^{1/x^2} is a hyperfunction solution of the hypoelliptic equation $x^3u' + 2u = 0$, but the hypoelliptic equation $x^3u' - 2u = 0$ has no non-C^∞ hyperfunction solutions with supp $u \neq \{0\}$). It is easy to see that there exists no non-smooth hyperfunction solution of $Lu = 0$ with $\operatorname{supp} u \neq \{0\}$ if and only if $\operatorname{Re} Q_j(x) \to -\infty$ as $x \to 0$ for $r < j \leqq m$ (see Section 3). If condition (S) is satisfied, then this condition can be expressed in terms of zeros of $p(x, \zeta)$ (see Section 6). On the other hand, there always exists a hyperfunction solution of $Lu = 0$ with $\operatorname{supp} u = \{0\}$, if the origin is an irregular singular point. Hence, all hyperfunction solutions of an ordinary differential operator L are smooth if and only if L is elliptic.

Note that an ordinary differential operator with analytic coefficients is analytic-hypoelliptic only if it is elliptic (unlike certain partial differential operators which are analytic-hypoelliptic without being elliptic). This is obviously true for operators L which admit C^∞ non-analytic solutions for the homogeneous equation $Lu = 0$. In other cases, where all distribution solutions of the homogenous equation are analytic (e.g., for $Lu = x^3u' + 2u$), it follows from the arguments of section 3 that there exist analytic functions f such that the equation $Lu = f$ possesses a smooth solution $u \in C^\infty$, but u is not analytic.

We remark that an ordinary differential operator is locally solvable at a point where it is hypoelliptic (given that $a_m(x)$ does not vanish to infinite order). This follows either from the method of proof of Theorem 1 or from the fact (also mentioned in section 3) that the formal adjoint of L satisfies the conditions of Theorem 1 if these conditions are satisfied for L. Thus, the phenomenon exemplified in [10] cannot occur in our present situation.

We hope to return the questions of the "explicit" determination of the determining factors and the verification of condition (ii) of Theorem 1 at another time. We remark in passing that, thus far, we have succeded in doing this for the most general second order equation.

I am very much indebted to Professors B. Malgrange and B. Helffer for bringing to my attention an error in an earlier version of this paper, one in which the importance of the condition (S) was overlooked, and to Professor S. Agmon for suggesting that the theorems remain true for equations with C^∞ coefficients, rather than analytic coefficients.

2. Structure of solutions near singular points

In this section, we summarize, in a form suitable for our purposes, several essentially well known results about the structure of the solutions of an ordinary differential equation in the neighborhood of a singular point. We shall assume, unless otherwise stated, that not all the coefficients of the operator $Lu = \sum_{j=0}^{m} a_j(x) d^j u / dx^j$ vanish at the singular point, which we take to be the origin. We shall also assume that the functions $a_j(x)$, $0 \leqq j \leqq m$, are C^∞ in a neighborhood of the origin. Let n_j be the order of the zero of $a_j(x)$ at the origin. We assume throughout that $n_m < \infty$. Suppose first that the functions $a_j(x)$ are holomorphic near $x = 0$. Then the symbol $p(x,\xi) = \sum_{j=0}^{m} a_j(x)(i\xi)^j$ can be factorized, by means of Puiseux series, as

$$p(x,\xi) = c(x)x^{m_n} \prod_{j=1}^{m} \Big(\xi - \sum_{k=N(j)}^{\infty} \alpha_{j,k} x^{k/q}\Big) \tag{2.1}$$

where q is a positive integer, $N(j)$ is a finite (positive or negative) integer, $c(x) = a_m(x)/x^{m_n}$ is non-zero in a neighborhood of the origin, $\alpha_{j,N(j)} \neq 0$ unless $\alpha_{j,k} = 0$ for all k, and the series converge in a (maybe punctured) neighborhood of the origin (see e.g. [4, p. 275]). If the functions $a_j(x)$ are only assumed to be C^∞ near $x = 0$, then each $a_j(x)$ gives rise to the formal power series

$$\sum_{n=0}^{\infty} \frac{(a_j^{(n)}(0))}{n!} x^n,$$

and (2.1) holds in the sense of formal power series; the series $\sum_{k=N(j)}^{\infty} \alpha_{j,k} x^{k/q}$ are formal fractional power series, and do not converge, in general.

Recall that the characteristic index [1] or the class [2] of the operator is defined to be $m - r$, where the integer r, $0 \leqq r \leqq m$, is specified by the requirements

$$\begin{aligned} n_j - j &> n_r - r \quad \text{for } j > r \\ n_j - j &\geqq n_r - r \quad \text{for } j < r. \end{aligned} \tag{2.2}$$

Set $\beta(j) = N(j)/q$ if $\alpha_{j,N(j)} \neq 0$ and $\beta(j) = \infty$ if $\alpha_{j,k}$ vanishes for all k. We shall assume, from now on, without explicitly mentioning it, that the factors in (2.1) are labelled in such a way that the sequence $\beta(j)$ is non-increasing. We claim that $\beta(r) \geqq -1$, and $\beta(r+1) < -1$. By equating the coefficients (in (2.1)) of $x^{n_j}\xi^j$ we see immediately that $n_j \geqq n_m + \sum_{k=j+1}^{m} \beta(k)$, and that if $\beta(j) > \beta(j+1)$ then $n_j = n_m + \sum_{k=j+1}^{m} \beta(k)$. Let ρ be defined by $\rho = \max\{j : \beta(j) \geqq -1\}$. Then $\beta(\rho+1) < -1$ which implies $n_\rho = n_m + \sum_{k=\rho+1}^{m} \beta(k)$. For $j > \rho$,

$$n_j - n_\rho \geqq \sum_{k=j+1}^{m} \beta(k) - \sum_{k=\rho+1}^{m} \beta(k) = -\sum_{k=\rho+1}^{j} \beta(k) > j - \rho.$$

For $j < \rho$,

$$n_j - n_\rho \geqq \sum_{k=j+1}^{m} \beta(k) - \sum_{k=\rho+1}^{m} \beta(k) = \sum_{k=j+1}^{\rho} \beta(k) \geqq j - \rho.$$

Hence $n - \rho$ is the characteristic index and $\rho = r$.

The equation $Lu = 0$ possesses an indicial equation of order r. This indicial equation is obtained by equating to zero the coefficient of the lowest order term in the expansion of $x^{-\rho} L(x^\rho)$ (this formal expansion exists not only in the analytic case, but also in the C^∞ case). It is clear that the lowest power of x in that expansion is $n_r - r$, and that the functions $a_j(x)$ for $j > r$ have no relevancy for the indicial equation. Note also that for $r = 0$, the indicial equation has no roots, and that $r = m$ corresponds (in the analytic case) to a regular singular point at the origin.

Using the roots of the indicial equations, one obtains (by equating coefficients) r distinct formal power series solutions of $Lu = 0$; if several roots coincide or differ by integers, then formal log-power series are obtained. If $r < m$ or if the coefficients are not analytic, then those formal power series do not necessarily converge. It can be shown, however, that they represent, asymptotically, r linearly independent actual solutions of $Lu = 0$. This last fact is proved explicitly in the literature [7, 8] only under the additional assumption that $a_j(z)$ are analytic for $z = re^{i\theta}$, $|r|, |\theta| < \varepsilon$, $\varepsilon > 0$, and will be used only under this assumption.

Turning now to the remaining $m - r$ solutions of the homogeneous equation, we recall that in the classical theory one looks for functions $Q(x)$ of the form

$$Q(x) = \sum_{k=1}^{s} \gamma_k x^{-k/q}$$

(here q is not necessarily the same as in the Puiseux expansion (2.1); rather, it is a multiple of the integer occurring in (2.1)) such that the linear differential operator M defined by $Mv = e^{-Q} L(e^Q v)$ has a characteristic index which is strictly less than m. Such a function $Q(x)$ is called a "determining factor". (Note that (2.2) and the discussion following it are valid also if $a_j(x)$ are expandible in fractional power series; in that case, the solutions are given as log fractional power series.) Let $Q(x)$ be such that the characteristic index of M is $m - j < m$, so that the equation $Mv = 0$ possesses an indicial equation of order $j \geqq 1$. Hence j formal (log) fractional power series expansions exist for v, and it is shown in classical texts (e.g., in [7]), that the resulting series for $u = e^{Q}v$ represent asymptotically as $x \to 0$, actual solutions of $Lu = 0$. Moreover, using all possible determining factors, we obtain $m - r$ linearly independent solutions of $Lu = 0$. In this manner, the existence of m linearly independent solutions (1.1)–(1.3) is established in the classical theory. (Once again, the $m - r$ linearly independent formal solutions exist always; the existence of actual solutions is proved in the literature under the additional hypothesis that $a_j(z)$ are also holomorphic for $z = re^{i\theta}$, $0 < |r| < \varepsilon$, $|\theta| < \varepsilon$, for ε small enough.)

The Puiseux expansion (2.1) gives us a simple expression for the determining factors under the following restriction. We say that the symbol $p(x,\xi)$ satisfies the condition (S') if whenever $-1 > \beta(i) = \beta(j)$ but $i \neq j$, then $\alpha_{i,N(i)} \neq \alpha_{j,N(j)}$. In fact, if the condition (S') is satisfied, then the derivative of the determining factor Q_j is given by the following simple formula:

$$\frac{dQ_j}{dx} = i \sum_{k=N(j)}^{-q-1} \alpha_{j,k} x^{k/q} \qquad j = r+1,\cdots,m. \tag{2.3}$$

In order to verify (2.3), rewrite M, using Leibnitz's rule and setting $e^{-Q}(e^{Q})^{(n)} = S(n,x)$, as

$$\begin{aligned} Mv &= e^{-Q}L(e^{Q}v) = \sum_{j=0}^{m} a_j(x) \sum_{h=0}^{j} \binom{j}{h} e^{-Q}(e^{Q})^{(j-h)}v^{(h)} \\ &= \sum_{h=0}^{m}\left(\sum_{j=h}^{m}\binom{j}{h}a_j(x)S(j-h,x)\right)v^{(h)}(x). \end{aligned} \tag{2.4}$$

Note that $S(0,x) \equiv 1$ and $S(n+1,x) = S'(n,x) + S(n,x)Q'(x)$. Setting now $Q'(x) = \sum_{k=q+1}^{s} c_k x^{-k/q}$, we see immediately (by induction) that

$$S(n,x) = [Q'(x)]^n + R_n(x)$$

where $R_n(x)$ is a polynomial in $x^{-1/q}$ of degree not exceeding $(n-1)s + q$. Hence the lowest $s - q$ terms in the coefficient of v in M appear in $\sum_{j=0}^{m} a_j(x)(Q'(x))^j$. Set $\min_{0 \leqq t \leqq m} qn_t - st = w$ and set $K = \{t: 0 \leqq t \leqq m,\ qn_t - st = w\}$.

Then the lowest term in the coefficient of v in M is the term of order w/q appearing in $\sum_{t \in K} a_t(x)(Q'(x))^t$, and the lowest term in the coefficient of v' is the term of order $(w+s)/q$ in $\sum_{t \in K} t a_t(x)(Q'(x))^{t-1}$. If condition (S') is satisfied, $s = -N(j) - q$ and $Q'(x)$ is given by (2.3), then the value of n_1 for M will be exactly equal to $(w+s)/q$, whereas the lowest $(s-q)$ terms in the coefficient of v will all vanish. Thus the value of n_0 for M is at least $w/q + (s-q)/q = n_1 - 1$. Hence $r \geqq 1$ for M and Q is indeed a determining factor.

If condition (S') is violated, then other (much more complicated) methods have to be employed in order to calculate the complete determining factors; only the leading (lowest) term of the determining factor is given, in the general case, by means of (2.3) (this was pointed out to the author by B. Malgrange and B. Helffer). As mentioned earlier we hope to return to this question in the future, and remark here only that the second order case has been settled completely.

Note that if the functions $a_j(x)$ are holomorphic for $x \neq 0$ in a sufficiently small sector, so that the formal log-power series represent asymptotically an actual solution u of $Lu = 0$, one may differentiate the series to an abitrarily high order and obtain an asymptotic representation of the corresponding derivative of u. While it is proved explicitly in [7] that one might differentiate the asymptotic series $m - 1$ times, we see, considering also the equation (of order $m + k$) $(d^k/dx^k)Lu = 0$, that we may differentiate up to the order $m + k - 1$, with k arbitrary.

3. Proof of sufficiency in Theorem 1

Let us assume, to begin with, that the coefficients $a_j(x)$ of L are C^∞ functions in a neighborhood U of the origin and are holomorphic functions of z in a sector $z = re^{i\theta}$, $0 < |r| < \varepsilon$, $|\theta| < \varepsilon$, for ε small enough. Then the structure theory, described in Section 2, applies ([7], [8]), and (1.3) is valid asymptotically. (The analyticity assumption will be removed at the end of this section.)

Assumption (i) of Theorem 1 and (2.2) imply that $a_r(0) \neq 0$, $a_j(0) = 0$ for $j > r$. Thus the indicial equation of L is $a_r(0)\rho(\rho - 1)\cdots(\rho - r + 1) = 0$; its roots are $\rho = 0, 1, \cdots, r - 1$. No logarithmic terms occur here (even though the roots differ by positive integers) and the resulting r formal power series represent asymptotically r linearly independent C^∞ functions $u_i(x)$, $1 \leqq i \leqq r$, in a neighborhood of the origin.

Assumption (ii) of Theorem 1 implies that for every non-vanishing determining factor $Q_i(x)$ $(r < i \leqq m)$, either $\operatorname{Re} Q_i(x) \to +\infty$ as $x \to 0_+$, or $\operatorname{Re} Q_i(x) \to -\infty$ as $x \to 0_+$. If for some i, $r < i \leqq m$, $u_i(x) = e^{Q_i(x)}v_i(x)$ (where $v_i(x)$ is represented asymptotically by a (log $-$) power series — see (1.2) and (1.3)) is the restriction of a distribution $u \in \mathscr{D}'(-\varepsilon, \varepsilon)\,(\varepsilon > 0)$ with $Lu = 0$, then $\operatorname{Re} Q_i(x) \to -\infty$. Otherwise u_i is the restriction of a hyperfunction solution, but not of a distribution. In fact, it is easy to construct a sequence $\phi_n(x) \in C_0^\infty(0, \varepsilon)$ with $\|\phi_n\|_s \leqq C(s)n^{a(s)}$ for all real s, but $|\int e^{Q_i(x)}v_i(x)\phi_n(x)dx| > e^{n^\delta}$ for some positive δ. If $\operatorname{Re} Q_i(x)$

$\to -\infty$, then $e^{Q_i(x)}$, and consequently $u_i(x)$ (as well as all the derivatives of $u_i(x)$) has a zero of infinite order at $x = 0_+$. A similar argument shows that $u_i(0_-) = u_i^{(1)}(0_-) = \cdots = u_i^{(n)}(0_-) = \cdots = 0$. Hence all distribution solutions u_i of $Lu = 0$ (which are represeented in (1.1)–(1.3)) are C^∞ functions in a neighborhood of the origin. (Caution: a distribution solution might vanish identically for $x < 0$ and be non-zero for $x > 0$, e.g., $Lu = x^2u' - u$.) Note that the same argument shows that there exists a non-smooth hyperfunction solution of $Lu = 0$ with $\operatorname{supp} u \neq \{0\}$ if and only if $\operatorname{Re} Q_i(x) \to +\infty$ for $x \to 0_+$ or $x \to 0_-$, for at least one index $r < i \leqq m$.

We claim now that all distribution solutions of $Lu = 0$ are C^∞ near the origin. We still have to prove that if $u = \sum_{i=1}^m c_i H(x) u_i(x) + \sum_{i=0}^k d_i \delta^{(i)}(x)$ is a distribution solution of $Lu = 0$, then $c_i = 0$ for $1 \leqq i \leqq r$ and $d_i = 0$ for $0 \leqq i \leqq k$ ($H(x)$ is the Heaviside function, $H(x) \equiv 1$ for $x > 0$ and $H(x) \equiv 0$ for $x < 0$). This, however, follows immediately from the non-vanishing of $a_r(0)$. In fact, the relations

$$L(H(x)x^j) = H(x)Lx^j + a_r(0)\delta^{(r-j-1)}(x) + \sum_{r \geqq i > j} e_{i,j}\delta^{(r-i-1)}(x)$$

and

$$L(\delta^{(j)}(x)) = a_r(0)\delta^{(r+j)}(x) + \sum_{i<r} f_{i,j}\delta^{(i+j)}(x)$$

imply, first of all, that $d_k a_r(0)\delta^{(r+k)}(x)$ cannot be cancelled by other terms of Lu, so that $d_k = 0$. Repeating the argument, we see that all the d_i's vanish. Now $c_1 a_r(0)\delta^{(r-1)}$ cannot be cancelled by any other term; hence $c_1 = 0$, and so forth. (Here we have set $u_i(x)$ to be the solution corresponding to the root $\rho = i - 1$ of the indicial equation, $1 \leqq i \leqq r$.)

It remains to show that for every $f \in C^\infty(U)$ there exists a C^∞ function u, defined in a neighborhood of the origin, such that $Lu = f$. Setting $x = 0$ in the differential equation, we get

$$a_r(0)u^{(r)}(0) + \sum_{j=0}^{r-1} a_j(0)u^{(j)}(0) = f(0).$$

By Leibnitz's rule,

$$\frac{d^k}{dx^k}\, a_j(x)u^{(j)}(x))\big|_{x=0} = \sum_{l=n_j}^{k} \binom{k}{l} a_j^{(l)}(0)u^{(j+k-l)}(0).$$

Let $m \geqq j > r$. Since $n_j > j - r$ for these values of j by (2.2) and assumption (i), we see that $j + k - l \leqq j + k - n_j < k + r$. Hence we obtain, by differentiating $Lu = f$ k times and equating to zero, that

$$a_r(0)u^{(k+r)}(0) + \sum_{s \leqq k+r-1} c_{k,s}u^{(s)}(0) = f^{(k)}(0) \tag{3.1}$$

where $c_{k,s}$ are constants. Thus we can find (choosing $u(0), u^{(1)}(0), \cdots, u^{(r-1)}(0)$ arbitrarily) a sequence $\{u^{(j)}(0)\}_{j=0}^{\infty}$ which satisfies the infinite triangular system (3.1) of linear equations. It is well known that for every infinite sequence $\delta_0, \delta_1, \cdots, \delta_n, \cdots$, there exists a function $v \in C_0^{\infty}(R^1)$ with $v^{(n)}(0) = \delta_n$ for $n = 0, 1, 2, \cdots$. Setting $\delta_n = u^{(n)}(0)$ we obtain a function $v \in C_0^{\infty}(R^1)$ having the property that the C^{∞} function g defined by $g = Lu - f$ satisfies $g^{(n)}(0) = 0$, $n = 0, 1, 2, \cdots$. (g is defined only in a neighborhood of the origin). Hence we have to consider only the inhomogeneous equations $Lu = g$ where $g^{(n)}(0) = 0$ for all $n \geqq 0$.

Recall the well known variation of constants formula:

$$u(x) = \sum_{j=1}^{m} c_j(x)u_j(x) \tag{3.2}$$

where $u_1(x), \cdots, u_m(x)$ are linearly independent solutions of $Lu = 0$, and the coefficients $c_j(x)$ satisfy the relation

$$\frac{dc_j}{dx} = \frac{\Delta_j(x)}{\Delta(x)}\, g(x) \tag{3.3}$$

where Δ is the Wronskian determinant $W(u_1, \cdots, u_m)$, and Δ_j is the algebraic complement (cofactor) of $u_j^{(m-1)}$ in the Wronskian. Let $u_1, \cdots, u_m$ be the solutions represented asymptotically by (1.1)–(1.3). It is proved in [7] that

$$\frac{\Delta_j(x)}{\Delta(x)} = e^{-Q_j(x)}w_j(x), \qquad j > r,$$

where $w_j(x)$ is analytic for $x \neq 0$ and $x^N w_j(x)$ is bounded near $x = 0$ for N large enough. Actually there is a gap in the proof given in [7], since the possibility of $r > 0$ (in our notation) is completely overlooked. Nevertheless, the argument of [7] can easily be modified so as to take care of that case also. We now observe that L^*, the formal adjoint of L, has the same characteristic index $m - r$ (see e.g. [1, 2]; it is even possible to show directly that the determining factors of L^* are minus the complex conjugates of those of L). Moreover, it is well known (and it follows easily from the variation of constants formula) that the complex conjugates of the functions $\Delta_j(x)/\Delta(x)$, $j = 1, \cdots, m$ form a linearly independent system of solutions of $L^*w = 0$. There are exactly $m - r$ linearly independent solutions of $L^*w = 0$ which are not representable, even asymptotically, as sums of formal log-power series. Hence those solutions are precisely the complex conjugates of Δ_j/Δ, $r+1 \leqq j \leqq m$, which proves that $w_j(x)$ is expressible asymptotically as the sum of a formal log-power series, and that there exist constants $M(k)$ such that $x^{M(k)}(\Delta_j(x)/\Delta(x))^{(k)}$ is bounded for $x \to 0$, $k = 0,1,\cdots, 1 \leqq j \leqq r$.

It clearly suffices to prove the existence of a solution u (of the equation $Lu = g$) defined for $\varepsilon > x > 0$, such that for all k, $\lim_{x\to 0_+} u^{(k)}(x)$ exists and is equal to zero. Set

$$(3.4)\qquad c_j(x) = \begin{cases} \displaystyle\int_0^x \frac{\Delta_j(t)}{\Delta(t)}\, g(t)dt & \text{if } j \leqq r \text{ or if } j > r \text{ and } \operatorname{Re} Q_j(x) \to +\infty,\ x \to 0_+ \\ \displaystyle\int_x^\varepsilon \frac{\Delta_j(t)}{\Delta(t)}\, g(t)dt & \text{if } j > r \text{ and } \operatorname{Re} Q_j(x) \to -\infty,\ x \to 0_+ \end{cases}$$

It is then clear that $c_j(x)u_j(x)$ tends to zero as $x \to 0_+$, and the same holds for derivatives of all orders of $c_j(x)u_j(x)$, for $1 \leqq j \leqq r$. Set $\operatorname{Re} Q_j(x) = R_j(x)$. If $R_j(x) \to +\infty$ as $x \to 0_+$, then

$$c_j(x)u_j(x) = v_j(x)e^{i\operatorname{Im}Q_j(x)}e^{R_j(x)}\int_0^x w_j(t)e^{-R_j(t)}g(t)e^{-i\operatorname{Im}Q_j(t)}\,dt.$$

Each of the functions $\operatorname{Re} c_j(x)u_j(x)$ and $\operatorname{Im} c_j(x)u_j(x)$ is a linear combination with constant coefficients of terms of the form $\phi(x) = P(x)e^{R_j(x)}\int_0^x e^{-R_j(t)}h(t)dt$ where (1) $h(t) \in C^\infty[0,\varepsilon)$, $h^{(k)}(0_+) = 0$ for all $k \geqq 0$ and $h(t)$ is real for $0 \leqq t < \varepsilon$; and (2) for every n there exists a constant $m(n)$ such that $x^{m(n)}P^{(n)}(x)$ is bounded as $x \to 0_+$. Let k be an arbitrarily large positive integer. By L'Hopital's rule,

$$\lim_{x\to 0+}\frac{\phi(x)}{P(x)x^k} = \lim_{x\to 0+}\frac{\displaystyle\int_0^x e^{-R_j(t)}h(t)dt}{x^k e^{-R_j(x)}}$$

$$= \lim_{x\to 0+}\frac{e^{-R_j(x)}h(x)}{e^{-R_j(x)}(kx^{k-1} - R_j'(x)x^k)} = 0,$$

since $R_j(x)$ is an algebraic function. In particular, $\phi(x) \to 0$ as $x \to 0_+$. Differentiating once, we get

$$\phi'(x) = (P'(x) + R_j'(x)P(x))e^{R_j(x)}\int_0^x e^{-R_j(t)}h(t)dt + P(x)h(x).$$

The same argument (use of L'Hopital's rule) applies to the first term, and the second term has a zero of order infinity. Thus $\lim_{x\to 0_+} \phi'(x) = 0$, and so on. If $R_j(x) \to -\infty$ as $x \to 0_+$ then

$$c_j(x)u_j(x) = v_j(x)e^{i\mathrm{Im}Q_j(x)}e^{R_j(x)}\int_x^\varepsilon w_j(t)e^{-R_j(t)}e^{-i\mathrm{Im}Q_j(t)}g(t)dt$$

and we have to consider terms of the form

$\psi(x) = P(x)e^{R_j(x)}\int_x^\varepsilon e^{-R_j(t)}h(t)dt,$ where h and P satisfy the same conditions 1) and (2) as above. Consider now the expression

$\dfrac{\psi(x)}{P(x)x^k} = \dfrac{\int_x^\varepsilon e^{-R_j(t)}h(t)dt}{e^{-R_j(x)}x^k}$. If the numerator does not tend to infinity as $x \to 0_+$, we are through. Otherwise, we apply L'Hopital's rule once again and obtain $\lim_{x\to 0_+} \dfrac{\psi(x)}{P(x)x^k} = 0$ and then continue for the higher derivatives.

If the functions $a_j(x)$ are only assumed to be smooth near $x = 0$ (and $n_m < \infty$), we construct, (as is possible according to [8]) C^∞ functions $b_j(x)$ which are analytic for $x = re^{i\theta}$, $0 < |r| < \varepsilon$, $|\theta| < \varepsilon$ for ε small enough and which moreover satisfy the equations $a_j^{(k)}(0) = b_j^{(k)}(0)$ for all $k \geqq 0$. The operator

$$M = \Sigma b_j(x)d^j/dx^j$$

satisfies the conditions of Theorem 1 if and only if L does, and the desired conclusion will follows from

LEMMA 3.1. *Let* $Lu = \Sigma_{j=0}^m a_j(x)d^ju/dx^j$, $Mu = \Sigma_{j=0}^m b_j(x)d^ju/dx^j$ *be differential operations with* C^∞ *coefficients in a neighborhood of the origin, such that* $a_j^{(k)}(0) = b_j^{(k)}(0)$ *for* $k = 0,1,2,\cdots$, $0 \leqq j \leqq m$. *Let* $a_m(0) = b_m(0) = 0$, *but* $a_m^{(K)}(0) = b_m^{(K)}(0) \neq 0$ *for some positive* K. *Then* L *is hypoelliptic near the origin if and only if* M *is hypoelliptic near the origin.*

PROOF. We restrict our attention to a neighborhood $(-\varepsilon,\varepsilon)$ of the origin, in which $a_m(x)$ or $b_m(x)$ vanish only if $x = 0$. Assume that M is hypoelliptic near the origin, and let $u \in \mathscr{D}'(-\varepsilon,\varepsilon)$ be such that $Lu \in C^\infty(-\varepsilon,\varepsilon)$. But L is elliptic for $x \neq 0$, $|x| < \varepsilon$. Hence $u \in C^\infty((-\varepsilon,0)\cup(0,\varepsilon))$. There exists a continuous function $v \in C[-\varepsilon/2, \varepsilon/2]$ and an integer n such that $d^n v/dx^n = u$ in $(-\varepsilon/2, \varepsilon/2)$ [4, p. 8]. Thus $|v(x)| \leqq C$ for $|x| \leqq \varepsilon/2$ and some constant C, and

$$\sum_{j=n}^{m+n} a_{j-n}(x) \frac{d^j v}{dx^j} = f.$$

We shall demonstrate, more generally, that if $v \in C^\infty((0,\varepsilon/2) \cup (-\varepsilon/2, 0)) \cup C[-\varepsilon/2, \varepsilon/2]$ is a solution of the equation

$$v^{(N)} = \sum_{j=0}^{N-1} c_j(x) \frac{d^j v}{dx^j} + g(x) \tag{3.5}$$

for $0 < |x| < \varepsilon/2$, where for every positive integer k there exist constants $D(k)$ and $m(k)$ such that

$$|c_j^{(k)}(x)|, \ |g^{(k)}(x)| \leqq D(k)|x|^{-m(k)} \quad \text{for } 0 < |x| < \varepsilon,$$

then for every positive integer k there exist constants $C(k)$ and $n(k)$ such that

$$|v^{(k)}(x)| \leqq C(k) x^{-n(k)} \quad 0 < |x| < \varepsilon/2. \tag{3.6}$$

Clearly, it suffices to prove (3.6) for $0 < x < \varepsilon/2$. Let x_0 be any number in $(0, \varepsilon/2)$, and let $\phi \in C^\infty(R^1)$ be such that $\phi(x) = 0$ for $x < 1$, $\phi(x) = 1$ for $x > 2$, $0 \leqq \phi(x) \leqq 1$ for $x \in R^1$, and set $\phi(x, a) = \phi(x/a)$ for $a > 0$. It follows from (3.5 that

$$\left[\phi\left(x, \frac{x_0}{3}\right) v\right]^{(N)} = \sum_{j=0}^{N-1} e_j(x) \frac{d^j v}{dx^j} + h(x)$$

where $e_j(x)$, $h(x) = 0$ for $x < x_0/3$, and for every positive integer k there exist constants $D(k,1)$ and $m(k,1)$ such that

$$|e_j^{(k)}(x)|, \ |h^{(k)}(x)| \leqq D(k,1) x_0^{-m(k,1)} \text{ for } 0 < x \leqq x_0. \tag{3.7}$$

Hence we obtain from integration by parts of the well-known formula

$$w(x) = \int_0^x \frac{(x-t)^{N-1}}{(N-1)!} w^{(N)}(t) dt + \sum_{j=0}^{N-1} w^{(j)}(0) \frac{x^j}{j!},$$

that

$$\phi\left(x, \frac{x_0}{3}\right) v(x) = \int_0^x v(t) \sum_{j=0}^{N-1} (-1)^j \frac{d^j}{dt^j} \left[\frac{e_j(t)(x-t)^{N-1}}{(N-1)!}\right] dt + \tag{3.8}$$
$$+ \int_0^x \frac{(x-t)^{N-1}}{(N-1)!} h(t) dt.$$

Differentiating (3.8) (with respect to x) for $2/3 x_0 < x \leqq x_0$, we see, taking (3.7) into account, that

$$|v'(x)| < C(1)x_0^{-n(1)} \text{ for } x \in [(2/3)x_0, x_0].$$

Differentiating (3.8) once more and using also the estimate for $v'(x)$, we see that $|v''(x)| < C(2)x_0^{-n(2)}$ for $x \in [(2/3)x_0, x_0]$. Iterating the procedure we obtain the estimate (3.6).

Using (3.6) for the function $v \in C[-\varepsilon/2, \varepsilon/2]$ where $d^n v/dx^n = u$ in $(-\varepsilon/2, \varepsilon/2)$ (in distribution sense), we see that for $\phi \in C_0^\infty(-\varepsilon/2, \varepsilon/2)$, $0 \leqq j \leqq m$,

$$[a_j(x) - b_j(x)] \frac{d^j u}{dx^j}(\phi) = (-1)^{n+j} \int_{-\varepsilon}^{\varepsilon} v(x) \frac{d^{n+j}}{dx^{n+j}}[(a_j(x) - b_j(x))\phi(x)]dx$$

$$= \int_{-\varepsilon}^{\varepsilon} v^{(n+j)}(x)[a_j(x) - b_j(x)]\phi(x)dx,$$

so that the distribution $[a_j(x) - b_j(x)]d^j u/dx^j$ is the C^∞ function $[a_j(x) - b_j(x)]\, d^{n+j}v/dx^{n+j}$ in $(-\varepsilon/2, \varepsilon/2)$, since the function $a_j(x) - b_j(x)$ along with its derivatives of all orders has a zero of infinite order at $x = 0$ for $0 \leqq j \leqq m$. Hence $Lu - Mu \in C^\infty(-\varepsilon/2, \varepsilon/2)$ and $Mu \in C^\infty(-\varepsilon/2, \varepsilon/2)$. The hypoellipticity of M implies that $u \in C^\infty$ in a full neighborhood of the origin.

4. Construction of certain distributions

In proving the necessity of the conditions of Theorem 1, we shall have to use certain distributions which are encountered frequently while solving homogeneous differential equations near singular points.

Recall the definition of the famous Heaviside function:

$$H(x) = \begin{cases} 1 & x \geqq 0 \\ 0 & x < 0 \end{cases}$$

For $\mathrm{Re}\,\alpha \geqq 0$, the function $H(\alpha, x) \overset{\text{def}}{=} x^\alpha H(x)$ is defined everywhere and it operates naturally on C_0^∞ functions (by integration) so that it is a distribution. Moreover, for $\mathrm{Re}\,\alpha > 1$, $dH(\alpha, x)/dx = \alpha H(\alpha - 1, x)$, both in the classical sense and in distribution sense. We now define for $\mathrm{Re}\,\alpha < 0$, α not an integer,

$$H(\alpha, x) \overset{\text{def}}{=} \frac{1}{(\alpha + 1)\cdots(\alpha + n)} \frac{d^n H}{dx^n}(\alpha + n, x)$$

where $n > -\mathrm{Re}\,\alpha$ and the differentiation is in distribution sense. It is clear that this definition is independent of n, and that $H(\alpha, x)$ is a continuous function for $x \neq 0$, $H(\alpha, x) = x^\alpha H(x)$ for $x \neq 0$.

Let now $n > -\mathrm{Re}\,\alpha$. Applying Leibnitz's rule (which is valid for the product of a C^∞ function and a distribution), we obtain

$$\frac{d^n}{dx^n}(xH(\alpha+n,x)) = x\,\frac{d^nH(\alpha+n,x)}{dx^n} + n\,\frac{d^{n-1}H(\alpha+n,x)}{dx^{n-1}}$$

$$= xH(\alpha,x)(\alpha+1)\cdots(\alpha+n) + nH(\alpha+1,x)(\alpha+2)\cdots(\alpha+n).$$

On the other hand,

$$\frac{d^n}{dx^n}(xH(\alpha+n,x)) = \frac{d^n}{dx^n}H(\alpha+n+1,x) = H(\alpha+1,x)(\alpha+2)\cdots(\alpha+n+1)\,.$$

Combining these, we see that

(4.1) $\quad x\,H(\alpha,x) = H(\alpha+1,x)$ for α not a negative integer.

We have to proceed somewhat differently in order to obtain suitable distributions from terms like $(\ln|x|)^m/x^k$. Observe first that

$$x\frac{d}{dx}(\ln|x|)^m = m(\ln|x|)^{m-1} \tag{4.2}$$

where m is a positive integer and the differentiation is to be understood in distribution sense. (Note that $(\ln|x|)^m$, a locally integrable function, is a legitimate distribution.) Indeed, for arbitrary $\phi\in C_0^\infty(R^1)$,

$$x\frac{d}{dx}(\ln|x|)^m(\phi) = \frac{d}{dx}(\ln|x|)^m(x\phi) = -(\ln|x|)^m((x\phi)')$$

$$= -\int_{-\infty}^{\infty}(\ln|x|)^m(x\phi'(x)+\phi(x))dx$$

$$= -\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\varepsilon} + \int_{\varepsilon}^{\infty}\right](\ln|x|)^m(x\phi'(x)+\phi(x))dx.$$

The last equation follows from Lebesgue's dominated integration theorem. Integrating by parts, we obtain

$$\int_\varepsilon^\infty(\ln|x|)^m x\phi'(x)dx = x(\ln|x|)^m\phi(x)\big|_\varepsilon^\infty - \int_\varepsilon^\infty(\ln|x|)^m\phi(x)dx$$

$$- \int_\varepsilon^\infty m(\ln|x|)^{m-1}\phi(x)dx.$$

Using the dominated integration theorem once again, we see that

$$\int_0^\infty(\ln|x|)^m x\phi'(x)dx = -\int_0^\infty(\ln|x|)^m\phi(x)dx - \int_0^\infty m(\ln|x|)^{m-1}\phi(x)dx.$$

Similarly,

$$\int_{-\infty}^0(\ln|x|)^m\,x\phi'(x)dx = -\int_\infty^0(\ln|x|)^m\phi(x)dx - \int_{-\infty}^0 m(\ln|x|)^{m-1}\phi(x)dx.$$

Hence

$$x\frac{d}{dx}(\ln|x|)^m(\phi) = m(\ln|x|)^{m-1}(\phi).$$

We now define, as a first step, the regularization $R(x^{-k})$ of $1/x^k$ (k is a positive integer) to be the distribution $(-1)^{k-1}/(k-1)!\, d^k \ln|x|/dx^k$ (differentiation in distribution sense). Setting $m = 1$ in (4.2) we see that $xR(x^{-1}) = 1$. Applying Leibnitz's rule to this product of a C^∞ function and a distribution, we obtain

$$0 = \frac{d^{k-1}}{dx^{k-1}}(xR(x^{-1})) = x\frac{d^k}{dx^k}\ln|x| + (k-1)\frac{d^{k-1}}{dx^{k-1}}\ln|x|$$

$$= (-1)^k(k-1)!(R(x^{-k+1}) - xR(x^{-k})),$$

or $xR(x^{-k}) = R(x^{-k+1})$. Set also

$$R(x^{-1}(\ln|x|)^n) \overset{\text{def}}{=} \frac{1}{n+1}\frac{d}{dx}(\ln|x|)^{n+1}$$

for $n \geqq 0$, where the derivative is the distributional derivative.

Suppose now, inductively, that a regularization $R(x^j(\ln|x|)^n)$ of $x^j(\ln|x|)^n$ has been defined already for $0 \leqq n < m$ and j an arbitrary integer, or for $n = m$ and $j \geqq k$, in such a way that the relation

(a) $xR(x^j(\ln|x|)^n) = R(x^{j+1}(\ln|x|)^n)$

holds for $n < m$ and j arbitrary or $n = m$ and $j \geqq k$, and the relation

(b) $\dfrac{d}{dx}R(x^j(\ln|x|)^n) = nR(x^{j-1}(\ln|x|)^{n-1}) + jR(x^{j-1}(\ln|x|)^n)$

holds for $n < m$ and j arbitrary or $n = m$ and $j > k$, and the derivative is the distributional derivative. We define then

$$R(x^{k-1}(\ln|x|)^m) \overset{\text{def}}{=} \frac{1}{k}\left[\frac{d}{dx}R(x^k(\ln|x|)^m) - mR(x^{k-1}(\ln|x|)^{m-1}\right].$$

The right hand side is well defined, by the multiple induction. The condition (b) is automatically satisfied. To prove (a), we observe that by Leibnitz's rule,

$$\text{(4.3)}\qquad \frac{d}{dx}(xR(x^k(\ln|x|)^m) = x\frac{d}{dx}R(x^k(\ln|x|)^m) + R(x^k(\ln|x|)^m).$$

On the other hand, applying (a) in the case $n = m$, $j = k$, we see that

$xR(x^k(\ln|x|)^m) = R(x^{k+1}(\ln|x|)^m)$. Using (b) in the case $n = m$, $j = k+1$, we may thus rewrite (4.3) as

$$mR(x^k(\ln|x|)^{m-1}) + (k+1)R(x^k(\ln|x|)^m) = x\frac{d}{dx}R(x^k(\ln|x|)^m) + R(x^k(\ln|x|^m)\,.$$

The last equation can be rewritten (using (a) now for $n = m-1$ and $j = k$) as

$$x\frac{d}{dx}R(x^k(\ln|x|)^m) - mxR(x^{k-1}(\ln|x|)^{m-1}) = kR(x^k(\ln|x|)^m),$$

from which (a) follows in the case $n = m$, $j = k-1$.

Summing up, we conclude that we have found regularizations of the functions $x^\alpha H(x)$ (if α is not a negative integer) on one hand, and of $x^j(\ln|x|)^m$ (for j arbitrary integer, m non-negative integer) on the other hand, in such a manner that the formal rules of differentiation and of multiplication by integral powers of x carry over from the functions to the distributions.

We have also to find a suitable distribution $u_\lambda(x)$ such that $u_\lambda(x) = H(x)x^\lambda \exp(i\sum_{k=1}^{n} c_k/x^{d_k})$ for $x \neq 0$, where λ is an arbitrary complex number, c_k, d_k are real numbers, and $d_k > 0$, $1 \leqq k \leqq n$. Assume without loss of generality that $d_n > d_{n-1} > \cdots > d_1$. For $\mathrm{Re}\,\lambda \geqq 0$ there are no problems, since the function $x^\lambda \exp(i\sum_{k=1}^{n} c_k/x^{d_k})$ is bounded near the origin. Furthermore, the equation $xu_\lambda(x) = u_{\lambda+1}(x)$ holds for $\mathrm{Re}\,\lambda \geqq 0$. Suppose now that distributions $u_\lambda(x)$ have been defined for $\mathrm{Re}\,\lambda \geqq -j\,(d_n - d_{n-1})$ (j is a non-negative integer) in such a way, that the relations

(i) $u_\lambda(x) = H(x)x^\lambda \exp\left(i\sum_{k=1}^{n}\frac{c_k}{x^{d_k}}\right) \quad x \neq 0$

and

(ii) $xu_\lambda(x) = u_{\lambda+1}(x)$

hold for $\mathrm{Re}\,\lambda \geqq -j(d_n - d_{n-1})$. Set now, for $\mathrm{Re}\,\lambda < -j(d_n - d_{n-1})$,

$$(4.4)\quad u_\lambda(x) \overset{\text{def}}{=} -\frac{i}{c_n d_n}\left[(\lambda + d_n + 1)u_{\lambda+d_n}(x) - i\sum_{k=1}^{n-1} c_k d_k u_{\lambda+d_n-d_k} - \frac{d}{dx}\, u_{\lambda+d_n+1}\right]$$

(the differentiation is in distribution sense). This definition is legitimate for $\mathrm{Re}\,\lambda \geqq -(j+1)(d_n - d_{n-1})$, since for such values of λ,

$$\operatorname{Re}(\lambda + d_n - d_k) \geqq \operatorname{Re}(\lambda + d_n - d_{n-1}) \geqq -(j+1)(d_n - d_{n-1}) + d_n - d_{n-1}$$
$$= -j(d_n - d_{n-1})$$

and

$$\operatorname{Re}(\lambda + d_n),\ \operatorname{Re}(\lambda + d_n + 1) \geqq -j(d_n - d_{n-1}).$$

The relation (i) obviously holds for the new values of λ. Using the induction hypothesis and (4.4) we see that

$$xu_\lambda(x) = \frac{-i}{c_n d_n}\left[(\lambda + d_n + 1)u_{\lambda+d_n+1}(x) - i\sum_{k=1}^{n-1} c_k d_k u_{\lambda+d_n-d_k+1} - x\frac{d}{dx}u_{\lambda+d_n+1}\right]$$

(4.5)

$$= \frac{-i}{c_n d_n}\left[(\lambda + d_n + 2)u_{\lambda+d_n+1}(x) - i\sum_{k=1}^{n-1} c_k d_k u_{\lambda+d_n-d_k+1}(x) - u_{\lambda+d_n+1}(x) - x\frac{d}{dx}u_{\lambda+d_n+1}\right].$$

Application of Leibnitz's rule to the product $xu_{\lambda+d_n+1} = u_{\lambda+d_n+2}$ (this equation follows from the induction hypothesis) and insertion of the results in the right hand side of (4.5) yields

$$xu_\lambda(x) = -\frac{i}{c_n d_n}\left[(\lambda + d_n + 2)u_{\lambda+d_n+1}(x) - i\sum_{k=1}^{n-1} c_k d_k u_{\lambda+d_n-d_k+1}(x) - \frac{d}{dx}u_{\lambda+d_n+2}\right]$$
$$= u_{\lambda+1}(x),$$

the last equality following from the definition (4.4) of $u_{\lambda+1}(x)$ (note that (4.4) obviously holds for $\operatorname{Re}\lambda \geqq 0$).

In a similar manner, we can define a family $\tilde{u}_\lambda$ of distributions, in such a manner that

$$\tilde{u}_\lambda(x) = H(-x)x^\lambda \exp\left(i\sum_{k=1}^{n} c_k/x^{d_k}\right)$$

for $x \neq 0$, and (ii) and (4.4) are satisfied (with $\tilde{u}_\lambda$ replacing u_λ). This can be done of course, only if the real numbers d_k are such that $(-1)^{d_k}$ is real (for at least one branch).

5. Proof of necessity in Theorem 1

In proving the non-hypoellipticity (at the origin) of an operator L, whose only singularity is the origin we shall frequently use the following simple

PROPOSITION. *If L is hypoelliptic in an open set V containing the origin, then for all $k,n \geqq 0$ there exists a natural number N such that if $Lu \in C^N(V)$ and $u \in H^{loc}_{-k}(V)$ then $u \in C^n(V)$.*

As a matter of fact, $u \in C^{N+m}(V - \{0\})$ since L is elliptic in $V - \{0\}$. Set

$$H^{loc}_{-k}(V) \times C^\infty(V) \supset F = \{(u,v): Lu = v\}.$$

Then the projection $P: F \to H^{loc}_{-k}(V)$ is really a map $P: F \to C^\infty(V)$ due to the hypoellipticity of L. Appplying the closed graph theorem to P and noting that $\{0\}$ is a compact subset of V, we obtain the proposition. (Similar propositions are frequently used by L. Hörmander.)

Let us assume, to begin with, that all the coefficients $a_j(x)$ are C^∞ in a neighborhood of the origin and holomorphic in the sector $z = re^{i\theta}$, $0 < |r| < \varepsilon$, $|\theta| < \varepsilon$ for $\varepsilon > 0$, that $a_j(0) \neq 0$ for at least one index j, and that the multiplicity of the zero of $a_m(x)$ at $x = 0$ is finite. The conditions of Theorem 1 are not satisfied if either (i) $a_r(0) \neq 0$ or (ii) $Q_j(x)$ is purely imaginary in a (possibly one-sided) neighborhood of $x = 0$ for at least one index j, $r < j \leqq m$. ((i) and (ii) are not mutually exclusive.)

(i) In this case $n_r \geqq 1$. Recall that the operator L has an indicial equation or order r and, using the roots of this equation, we obtain r formal log-power series solutions of $Lu = 0$ (these solutions correspond to identically vanishing determining factors). The only difference between our present system of r such solutions and the system present in the case of a regular singular point of an equation of order r is that in our case the series need not converge; in most cases they only represent asymptotically r linearly independent solutions of $Lu = 0$ (assuming analyticity of the coefficients at $x = 0$ does not imply convergence). The formal theory of the series, however, parallels exactly the theory of the series obtained in the case of a regular singular point.

Let $\rho_1, \cdots, \rho_r$ be the roots (repeated according to their multiplicities) of the indicial equation, ordered by $\operatorname{Re}\rho_1 \geqq \operatorname{Re}\rho_2 \geqq \dots \geqq \operatorname{Re}\rho_r$. There exists a formal power series solution $u(x) \sim \sum_{j=0}^{\infty} c_j x^{\rho_1 + j}$. If ρ_1 is not an integer, then we set $u_n(x) = \sum_{j=0}^{n} c_j H(\rho_1 + j; x)$, and recalling that the distributions $H(\alpha, x)$ may be differentiated and multiplied by integral powers of x as if they were equal to x^α, we conclude that for every N there exists an integer n such that $Lu_n(x) \in C^N$. Moreover, $u_n(x) \in H^{loc}_{\rho_1}$ for all n. On the other hand, $u_n(x) \notin C(R^1)$ if $\operatorname{Re}\rho_1 < 0$, and $u_n(x) \notin C^{\operatorname{Re}\rho_1 + 1}(R^1)$ for $\operatorname{Re}\rho_1 \geqq 0$. It follows from the Proposition that L

cannot be hypoelliptic near $x = 0$. (If the series $\Sigma_{j=0}^{\infty} c_j x^{\rho_1 + j}$ converges, one may simply set $v(x) = \Sigma_{j=0}^{\infty} c_j H(j + \rho_1, x)$.)

If ρ_1 is a negative integer, we set

$$u_n(x) = \sum_{j < -\rho_1} c_j R(x^{j+\rho_1}) + \sum_{n \geqq j \geqq -\rho_1} c_j x^{\rho_1 + j}$$

and proceed as before. If ρ_1 is a non-negative integer, then $(H(x)w(x))^{(j)} = H(x)w^{(j)}(x)$ for $j \leqq \rho_1$, where $w(x)$ is the actual solution corresponding to the formal solution $u(x)$. If $\rho_1 \geqq m$ it follows that

$$L(H(x)w(x)) = H(x)Lw(x) = 0$$

whereas $H(x)w(x)$ is not infinitely differentiable near $x = 0$. Thus L is not hypoelliptic. If $r - 1 \leqq \rho_1 \leqq m - 1$, then we make use of the vanishing of $a_j(0)$ for $j \geqq r$. By Leibnitz's rule,

$$\begin{aligned}(H(x)w(x))^{(j)} &= H(x)w^{(j)}(x) + \sum_{i=1}^{j} \binom{j}{i} w^{(j-i)}(0)\delta^{(i-1)}(x) \\ &= H(x)w^{(j)}(x) + \sum_{i=1}^{j-\rho_1} \binom{j}{i} w^{(j-i)}(0)\delta^{(i-1)}(x).\end{aligned}$$

Hence

$$x^{n_j}(H(x)w(x))^{(j)} = x^{n_j}H(x)w^{(j)}(x) + \sum_{i=1}^{j-\rho_1} \binom{j}{i} w^{(j-i)}(0)x^{n_j}\delta^{(i-1)}(x).$$

If $j \leqq \rho_1$, then the sum disappears. If $j > \rho_1$, then, by (2.2),

$$n_j - (i-1) \geqq n_j - (j - \rho_1 - 1) \geqq n_r - r + \rho_1 + 1 \geqq n_r.$$

But in the present case (case (i)) $n_r \geqq 1$, so that $n_j - (i-1) \geqq 1$. Hence $a_j(x)(H(x)w(x))^{(j)} = H(x)a_j(x)w^{(j)}(x)$ for all j $(0 \leqq j \leqq m)$ and we obtain again $L(H(x)w(x)) = 0$.

If ρ_1 is a non-negative integer, $\rho_1 < r-1$, we turn our attention to ρ_2. If ρ_2 is not an integer, we can argue as before to show that L is not hypoelliptic. If ρ_2 is a negative integer, then a formal solution of the equation is given by

$$u(x) \sim x^{\rho_2} \sum_{j=0}^{\infty} c_j x^j + (\ln x) x^{\rho_1} \sum_{j=0}^{\infty} d_j x^j.$$

Then

$$\tilde{u}(x) \sim x^{\rho_2} \sum_{j=0}^{\infty} c_j x^j + (\ln|x|)x^{\rho_1} \sum_{j=0}^{\infty} d_j x^j$$

is also a formal solution, since $(\ln|x|)' = 1/x$. Setting

$$u_n(x) = \sum_{j<-\rho_2} c_j R(x^{j+\rho_2}) + \sum_{j=-\rho_2}^{n} c_j x^j + \sum_{j=0}^{n} d_j R(x^{\rho_1+j}\ln|x|)$$

and recalling the properties (a) and (b) of the distributions $R(x^k \ln|x|)$, we conclude that for every N there exists a sufficiently large n with $Lu_n \in C^N$. But $u_n \in H^{loc}_{\rho_2}(R^1)$ and u_n is discontinuous at the origin, so that L cannot be hypoelliptic. If ρ_2 is a non-negative integer and $\rho_2 = \rho_1$, then logarithmic terms are bound to occur. If $\rho_2 < \rho_1$ then the logarithmic terms might disappear, and we will have to consider ρ_3, and so forth. We will not be through after $r-1$ steps only if all the ρ_i are non-negative integers for $1 \leqq i \leqq r-1$, and $\rho_{r-1} < \cdots < \rho_2 < \rho_1 < r-1$; this is possible only if $\rho_{r-1} = 0$. Hence if $\rho_r = 0$, then logarithmic terms have to appear and L cannot be hypoelliptic. Assume, therefore, that $\rho_r < 0$. If ρ_r is not an integer we use the distributions $u_n(x) = \sum_{j=0}^{n} c_j H(\rho_r + j, x)$ where $\sum_{j=0}^{\infty} c_j x^{\rho_r + j}$ is a formal solution; otherwise we use the regularizations $R(x^k(\ln|x|)^n)$ for terms like x^{ρ_r+j} and for logarithmic terms which might turn out. We see, therefore, that L is hypoelliptic at $x = 0$ in no case, if (i) holds.

(ii) If $Q_j(x)$ is purely imaginary for $x > 0$ (the case where $Q_j(x)$ is purely imaginary only for $x < 0$ is treated similarly), then $Q_j(x) = i\sum_{k=1}^{n} c_k/x^{d_k}$ for $x > 0$, where c_k, d_k are real numbers, and $d_k > 0$, $1 \leqq k \leqq n$. There exist complex constants λ, γ_h, $h = 0, 1, \cdots$, such that

$$u \sim \exp\left[i \sum_{k=1}^{n} \frac{c_k}{x^{d_k}}\right] x^{\lambda} \sum_{h=0}^{\infty} \gamma_h x^{h/q}$$

is a formal solution of $Lu = 0$ (λ is taken to be the root with the greatest real part of the indicial equation associated with the determining factor $i\sum_{k=1}^{n} c_k x^{d_k}$, in order to avoid logarithmic terms). Set now $v_t(x) = \sum_{h=0}^{t} \gamma_h u_{h/q+\lambda}(x)$, where $u_\mu(x)$ is the distribution which regularizes $H(x)x^\mu \exp[i\sum_{k=1}^{n} c_k/x^{d_k}]$; this distribution was defined inductively in (4.4). Then $Lv_t \in C^N$ if t is large enough, $v_t \in H_s$ for $-s$ large enough (s is independent of t if t is large enough) but v_t is discontinuous if $\operatorname{Re}\lambda < 0$ and is not $(\operatorname{Re}\lambda + 1)$ times continuously differentiable if $\operatorname{Re}\lambda \geqq 0$. It follows from the proposition that L is not hypoelliptic. Note that we have to do some regularization even if $\mathrm{Re}\,\lambda \geqq 0$, since differentiation of the exponential might introduce negative powers of x.

Assume now that $a_j(0) = 0$ for all j, $0 \leqq j \leqq m$. Then, there exists a natural number n, $0 < n \leqq n_m$, such that $L = x^n\tilde{L}$, where $\tilde{L}u = \sum_{j=0}^{m} \tilde{a}_j(x)d^ju/dx^j$ is a differential operator whose coefficients satisfy the same smoothness and analyticity conditions as the $a_j(x)$, and $\tilde{a}_j(0) \neq 0$ for at least one j. If $\tilde{L}$ does not satisfy the conditions of Theorem 1, then, by the facts proved up to now, $\tilde{L}$ is not hypoelliptic near $x = 0$, i.e., there exists a distribution $u \in \mathscr{D}'(V)$ where V is a neighborhood of the origin, such that $\tilde{L}u \in C^\infty(V)$, but $u \notin C^\infty(V)$. Then $Lu = x^n\tilde{L}u \in C^\infty(V)$, which proves that L is not hypoelliptic near $x = 0$. If $\tilde{L}$ does satisfy the conditions of Theorem 1, we shall prove that there exists a distribution $u \in \mathscr{D}'(V)$ such that $\tilde{L}u = \delta$ (and therefore $u \notin C^\infty(V)$). Then $Lu = x^n\tilde{L}u = x^n\delta = 0$. Note first that the characteristic index $m - r$ of $\tilde{L}^*$ (the formal adjoint of $\tilde{L}$) is equal to that of $\tilde{L}$ (see e.g. [1, 2]). Moreover, it follows either from classical results (by rewriting $\tilde{L}$ and $\tilde{L}^*$ in their "normal" forms) or by direct calculation, that the coefficient of d^r/dx^r in $\tilde{L}^*$ does not vanish at $x = 0$. Also, the determining factors of $\tilde{L}^*$ satisfy condition (ii) of Theorem 1 if and only if the determining factors of $\tilde{L}$ do (see Section 3). Hence $\tilde{L}^*$ is hypoelliptic. It follows from a standard argument involving the Hahn-Banach theorem that $\tilde{L}u = \delta$ is solvable for $u \in \mathscr{D}'(V)$.

If the coefficients $a_j(x)$ are assumed only to be C^∞ near $x = 0$ (and $n_m < \infty$), we repeat the construction of the last paragraphs of Section 3 and apply Lemma 3.1. Thus L cannot be hypoelliptic if M is not hypoelliptic. But if L does not satisfy the conditions of Theorem 1, then M does not satisfy them and therefore (by the facts proved until now) it is not hypoelliptic.

6. Proof of Theorem 2

We begin by finding out what limitations on the orders of the zeros of the coefficients of L follow from the assumptions of Theorem 2.

LEMMA 6.1. *Let the coefficients $a_j(x)$ of the differential operator $L = \sum_{j=0}^{m} a_j(x)d^j/dx^j$ be analytic in a neighborhood U of the origin and let there exist a constant $C > 0$ such that for $x \in U$, ζ a complex number with $p(x,\zeta) = \sum_{j=0}^{m} a_j(x)(i\zeta)^j = 0$, either $|\zeta| < C$ or $|x|\,|\mathrm{Im}\,\zeta| \to \infty$ as $x \to 0$. Then $\beta(r) \geqq 0$, $\beta(r+1) < -1$, where $m - r$ is the characteristic index of L. Furthermore, $n_r = 0$.*

PROOF. It follows from the assumptions, that in the expansion (2.1), $0 > N(j) \geqq -q$ for no $0 \leqq j \leqq m$, since otherwise there would be unbounded branches of zeros of $p(x,\zeta) = 0$ for which $|x||\operatorname{Im}\zeta|$ does not tend to infinity for $x \to 0$. Recall that r is the largest index for which $\beta(r) = N(r)/q \geqq -1$. Hence in our case $\beta(r) \geqq 0$ and $\beta(r+1) < -1$. By the first inequality of (2.2), $n_j > n_r + (j-r) > n_r$ for $j > r$. For $j < r$,

$$n_j \geqq n_m + \sum_{i=j+1}^{m} \beta(i) = n_m + \sum_{i=r+1}^{m} \beta(i) + \sum_{i=j+1}^{r} \beta(i) = n_r + \sum_{i=j+1}^{r} \beta(i) \geqq n_r$$

since $\beta(i) \geqq 0$ for $i \leqq r$. Not all the coefficients $a_j(x)$ vanish at $x = 0$, since otherwise $p(0,\zeta) = 0$ for all complex ζ. Hence $n_j = 0$ for at least one index $0 \leqq j \leqq m$ and therefore $n_r = 0$; i.e., $a_r(0) \neq 0$. Note also that $n_m = -\sum_{j=r+1}^{m} \beta(j)$.

Thus we see that condition (i) of Theorem 1 is fulfilled if the conditions of Theorem 2 are satisfied. In the present case (analytic coefficients), condition (S) on the branches of roots of $p(x,\zeta)$ is equivalent to condition (S') on the Puiseux expansion. Hence formula (2.3) for the determination of the determining factors may be applied.

Let us fix the branch of $x^{1/q}$ which is positive for $x > 0$, with a cut at $\{z : \operatorname{Im} z \leqq 0,\ z \text{ purely imaginary}\}$. The assumption $|x||\operatorname{Im}\zeta| \to \infty$ as $x \to 0$ for the unbounded zeros of $p(x,\zeta) = 0$ implies that for every index $j, m \geqq j \geqq r+1$, there exists at least one index k (depending on j) with $N(j) \leqq k < -q$ and $\operatorname{Im}\alpha_{j,k} \neq 0$. Thus, by (2.3), every determining factor $Q_j(x)$, $r+1 \leqq j \leqq m$, satisfies the condition $|\operatorname{Re} Q_j(x)| \to \infty$ as $x \to 0_+$. By similar considerations we can also show that $|\operatorname{Re} Q_j(x)| \to \infty$ as $x \to 0_-$. Hence condition (ii) of Theorem 1 is also fulfilled.

Note that a similar argument shows that there exists a non-smooth hyperfunction solution of $Lu = 0$ with $\operatorname{supp} u \neq \{0\}$ for an operator L satisfying the assumptions of Lemma 6.1 and condition (S) if and only if either there exist indices j, l with $r < j \leqq m$ and $N(j) \leqq l < -q$ such that (i) $\operatorname{Im}\alpha_{j,l} < 0$ and (ii) $\alpha_{j,k}$ is purely real for $l < k < -q$, or a similar condition holds when one considers the coefficients of the Puiseux expansion which result if we choose a branch of $x^{1/q}$ which is real on the negative axis (compare Section 3).

We have thus established the sufficiency part of Theorem 2 for analytic coefficients. In order to treat other cases, we apply the following theorem, due to Ostrowski [9, p. 125]. Let

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad g(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n,$$

where a_i, b_i are complex. Set $\gamma = \frac{1}{2} \max_{1 \leqq i \leqq n} \left(\left| \frac{a_i}{a_0} \right|^{1/i}, \left| \frac{b_i}{b_0} \right|^{1/i} \right)$ and

$\varepsilon = \sqrt[n]{\sum_{i=1}^{n} \left| \frac{b_i}{b_0} - \frac{a_i}{a_0} \right| \gamma^{n-i}}$. Then in an ε-neighborhood of every root of $f(z)$,

there exists at least one root of $g(z)$ (and, of course, vice versa). Moreover, the roots $\{x_i\}$, $\{y_i\}$, $1 \leqq i \leqq n$, of $f(z)$ and $g(z)$ respectively, can be ordered in such a way that $|x_i - y_i| < 2n\varepsilon$, $1 \leqq i \leqq n$. We may now prove

LEMMA 6.2. *Let the functions $a_j(x)$, $0 \leqq j \leqq m$, be C^∞ in a neighborhood of the origin, and let the multiplicity n_m of the zero of $a_m(x)$ at $x = 0$ be finite. Let (2.1) be the formal Puiseux expansion associated with $p(x,\xi) = \sum_{j=0}^{m} a_j(x)(i\xi)^j$. Then there exist n functions $\phi_1(x), \cdots, \phi_m(x)$ which are bounded for x near the origin, such that*

$$p(x,\xi) = c(x) x^{n_m} \prod_{j=1}^{m} \left(\xi - \sum_{k=N(j)}^{-1} \alpha_{j,k} x^{k/q} + \phi_j(x) \right). \tag{6.1}$$

PROOF. Set $a_{j,N}(x) = \sum_{n=0}^{N} a_j^{(n)}(0) x^n / n!$, and consider $p_N(x,\xi) = \sum_{j=0}^{m} a_{j,N}(x)(i\xi)^j$. Applying Ostrowski's theorem to $f(z) = p(x,\xi)$ and $g(z) = p_N(x,\xi)$ we find that for N sufficiently large, $\varepsilon(x)$ can be taken to be bounded as $x \to 0$ and the roots $\xi_r(x)$, $\eta_r(x)$, $r = 1, \cdots, m$ of $p(x,\xi)$ and $p_N(x,\eta)$ can be ordered in such a way that $|\xi_r(x) - \eta_r(x)| < 2m\,\varepsilon(x)$, $1 \leqq r \leqq m$. For N large, the initial terms in the Puiseux expansion of $p_N(x,\xi)$ coincide with those of $p(x,\xi)$. Also, for every N, the series converge for $x \neq 0$. Hence, for N large enough,

$$p_N(x,\xi) = c(x) x^{n_m} \prod_{j=1}^{m} \left(\xi - \sum_{k=N(j)}^{-1} \alpha_{j,k} x^{k/q} + \psi_j(x) \right)$$

where $\psi_j(x)$ are analytic for $x \neq 0$ and bounded as $x \to 0$. Thus, $\eta_j(x) = \sum_{k=N(j)}^{-1} \alpha_{j,k} x^{k/q} - \psi_j(x)$. Hence (6.1) follows with

$$\phi_j(x) = \xi_j(x) - \eta_j(x) - \psi_j(x).$$

We can now generalize the arguments which were given at the beginning of this section.

LEMMA 6.1 bis. *Let the coefficients $a_j(x)$ of the differential operator $L = \sum_{j=0}^{m} a_j(x)d^j/dx^j$ be C^∞ in a neighborhood U of the origin, and let the multiplicity n_m of the zero of $a_m(x)$ at $x = 0$ be finite. Let there exist a constant $C > 0$ such that for $x \in U$, ζ a complex number with $p(x,\zeta) = \sum_{j=0}^{m} a_j(x)(i\zeta)^j = 0$, either $|\zeta| < C$ or $|x|\,|\mathrm{Im}\,\zeta| \to \infty\ x \to 0$. Then $\beta(r) \geqq 0$, $\beta(r+1) < -1$, where $m - r$ is the characteristic index of L, and $n_r = 0$.*

The proof of Lemma 6.1 bis parallels exactly that of Lemma 6.1, (6.1) replacing (2.1).

Thus we see that the condition (i) of Theorem 1 is satisfied if the assumptions of Theorem 2 are fulfilled. If condition (S) is satisfied, it follows from Lemma 6.2 that condition (S') is also satisfied, so that (2.3) can be applied. Applying Lemma 6.2 once again, we see that we can deduce from the assumptions of Theorem 2 the validity of condition (ii) of Theorem 1 in the C^∞ case, in the same way as we did above in the case of analytic coefficients. Thus, the sufficiency part of Theorem 2 is proved.

To prove the necessity of the conditions of Theorem 2 (if condition (S) holds) we note that those conditions are not satisfied if, in (2.1) (or (6.1)) either (i) $-q \leqq N(j) < 0$ for at least one index j, $1 \leqq j \leqq r$, or (ii) $\sum_{k=N(j)}^{-q-1} \alpha_{j,k} x^{k/q}$ is purely real in a (possibly one sided) neighborhood of $x = 0$ for at least one index j, $r + 1 \leqq j \leqq m$ ((i) and (ii) are not mutually exclusive).

(i) In this case $0 > \beta(r)$, where $n - r$ is the characteristic index of L. All the coefficients $a_j(x)$ are finite at $x = 0$. Hence $n_j \geqq 0$ for $0 \leqq j \leqq m$. Let s be that index which satisfies $\beta(s) \geqq 0$, $\beta(s+1) < 0$. Then $n_s = n_m + \sum_{j=s+1}^{m} \beta(j)$ (equality holds here, since $\beta(s) > \beta(s+1)$). Hence

$$n_r = n_m + \sum_{j=r+1}^{m} \beta(j) = n_s - \sum_{j=s+1}^{r} \beta(j) > n_s \geqq 0,$$

so that $n_r \geqq 1$ or $a_r(0) = 0$. Thus, condition (i) of Theorem 1 is not fulfilled.

(ii) Since condition (S) is assumed to hold (so that by Lemma 6.2, condition (S') also holds) we may apply (2.3) and obtain at least one purely imaginary non-vanishing determining factor, so that condition (ii) of Theorem 1 is not satisfied.

Thus we have shown that the necessity of the conditions of Theorem 2 follows from Theorem 1.

7. Proof of Theorem 3

Let n_j be the multiplicity of the zero of $a_j(x)$ at $x = 0$. Then $a_j(x) = x^{n_j} b_j(x)$, where $b_j(x)$ is a C^∞ function near $x = 0$, and $b_j(0) \neq 0$. We claim that the conclusion (b) of Theorem 3 will follow once we prove that there exist constants C, δ, ρ, $0 \leqq \delta < \rho \leqq 1$ and a certain neighborhood V of the origin such that for α, β satisfying $n_j \geqq \beta \geqq 0$, $j \geqq \alpha \geqq 0$, there exist a constant $C(\alpha, \beta, V)$, such that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} x^{n_j} \xi^j \right| \leqq C(\alpha, \beta, V) \left| p(x, \xi) \right| (1 + |\xi|)^{\delta\beta - \rho\alpha} \tag{7.1}$$

for $x \in V$, ξ real, $|\xi| \geqq C$. Indeed, $\frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} a_j(x)\xi^j \equiv 0$ if $\alpha > j$, so that we may assume that $\alpha \leqq j$ and then $\partial^\alpha \xi^j / \partial \xi^\alpha = C(\alpha, j)\xi^{j-\alpha}$. For $\alpha \leqq j$,

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} a_j(x)\xi^j \right| = \left| C(\alpha, j)\xi^{j-\alpha} \frac{\partial^\beta}{\partial x^\beta} x^{n_j} b_j(x) \right|$$

$$= \left| C(\alpha, j)\xi^{j-\alpha} \sum_{k \leqq \min(\beta, n_j)} \binom{\beta}{k} C(k, n_j) x^{n_j - k} b_j^{(\beta-k)}(x) \right|$$

$$= \left| \sum_{k \leqq \min(\beta, n_j)} b_j^{(\beta-k)}(x) C(\alpha, \beta, j, k, n_j) \frac{\partial^{\alpha+k}}{\partial x^k \partial \xi^\alpha} x^{n_j} \xi^j \right|$$

$$\leqq |p(x,\xi)| \sum C'(\alpha, \beta, j, k, n_j)(1 + |\xi|)^{\delta k - \rho\alpha} \leqq C'' |p(x,\xi)| (1 + |\xi|)^{\delta\beta - \rho\alpha} .$$

This implies the required estimates for

$$\frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} p(x, \xi) = \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} \sum_{j=0}^{m} a_j(x)\xi^j i^j .$$

We note also that it suffices to prove the existence of an $\varepsilon > 0$ such that (7.1) holds whenever $0 < x < \varepsilon$, since we obtain the result, by considering $p(-x, \xi)$, for $-\varepsilon < x < 0$, and by continuity the result follows also for $x = 0$.

According to Lemma 6.2 we can expand $p(x, \xi)$ as

$$p(x, \xi) = c(x) x^{n_m} \prod_{j=1}^{m} \Big(\xi - \sum_{k=N(j)}^{-1} \alpha_{j,k} x^{k/q} + \phi_j(x)\Big) \tag{6.1}$$

where the functions $\phi_j(x)$ are bounded as $x \to 0$, and the constants $\alpha_{j,k}$ and $N(j)$ are the same as in the formal Puiseux expansion (2.1). We fix a branch of

$x^{1/q}$ which is positive for $x > 0$; this implies a specific choice of the constants $\alpha_{j,k}$. Let $m - r$ be the characteristic index of L. Lemma 6.1 bis can be applied, since its assumptions are certainly satisfied. Hence no negative powers appear in (6.1) for $j \leqq r$, and $\beta(j) = N(j)/q < -1$ for $j > r$. In particular we may rewrite (6.1) as

$$(7.2) \qquad p(x,\xi) = c(x)x^{n_m}\prod_{j=1}^{r}(\xi + \phi_j(x))\prod_{j=r+1}^{m}\left(\xi - \sum_{k=N(j)}^{-1}\alpha_{j,k}x^{k/q} + \phi_j(x)\right).$$

Observe that $\operatorname{Im}\alpha_{j,N(j)} \neq 0$ for $m \geqq j > r$ ($\alpha_{j,N(j)} \neq 0$ by definition), for otherwise $|\operatorname{Re}\zeta|$ will not be bounded by $C|\operatorname{Im}\zeta|$ on the "curve" $\zeta = \sum_{k=N(j)}^{-1}\alpha_{j,k}x^{k/q} + \phi_j(x)$, a curve which is contained in the set $\{(x,\zeta): p(x,\zeta) = 0\}$. Recall also that, according to Lemma 6.1 bis, $n_r = 0$. Hence $\sum_{j=r+1}^{m}\beta(j) = -n_m$, and we may rewrite (7.2) as

$$(7.3) \qquad p(x,\xi) = c(x)\prod_{j=1}^{r}(\xi + \phi_j(x))\prod_{j=r+1}^{m}(x^{-\beta(j)}\xi - \alpha_{j,N(j)} + \Psi_j(x))$$

where $\Psi_j(x) \to 0$ for $x \to 0$, $r+1 \leqq j \leqq m$.

We denote by C from now on a generic positive constant which is independent of the real variable ξ if $|\xi|$ is large enough, and which does not depend on x, if x is restricted to sufficiently small positive values. Note that for $r < j \leqq m$,

$$(7.4) \qquad |x^{-\beta(j)}\xi - \alpha_{j,N(j)} + \Psi_j(x)| \geqq C$$

$$(7.5) \qquad |x^{-\beta(j)}\xi - \alpha_{j,N(j)} + \Psi_j(x)| \geqq C|x^{-\beta(j)}\xi|.$$

As a matter of fact,

$$|\operatorname{Im}(x^{-\beta(j)}\xi - \alpha_{j,N(j)} + \Psi_j(x))| \geqq \tfrac{1}{2}|\operatorname{Im}(\alpha_{j,N(j)}|$$

if x is small enough, and

$$|x^{-\beta(j)}\xi| \leqq |\operatorname{Re}(x^{-\beta(j)}\xi - \alpha_{j,N(j)} + \Psi_j(x))| + |\operatorname{Re}(\alpha_{j,N(j)} - \Psi_j(x))|.$$

It follows from (7.3) and (7.4) that

$$(7.6) \qquad |p(x,\xi)| > C|\xi|^r$$

for $0 < x < \varepsilon$ and ξ large enough (C and ε are suitable constants), so that the conclusion (a) of Theorem 3 is proved. It follows also from (7.3), (7.4) and (7.5) that

$$|x^{-\sum_{j=r+1}^{l}\beta(j)}\xi^l| \leqq C|p(x,\xi)|$$

for $r+1 \leqq l \leqq m$, $0 < x < \varepsilon$ and ξ real, $|\xi|$ large. But

$$n_l \geqq n_m + \sum_{j=l+1}^{m} \beta(j) = -\sum_{j=r+1}^{m} \beta(j) + \sum_{j=l+1}^{m} \beta(j) = -\sum_{j=r+1}^{l} \beta(j),$$

so that $|x|^{n_l} < |x|^{-\Sigma_{j=r+1}^{l}\beta(j)}$ if $|x| < 1$. Hence

$$|x^{n_l}\xi^l| < C|p(x,\xi)|. \tag{7.7}$$

Let α, β be integers, $0 \leqq \alpha \leqq l$, $0 \leqq \beta \leqq n_l$. Then

$$\begin{aligned} \left| \frac{\partial^{\alpha+\beta} x^{n_l}\xi^l}{\partial\xi^\alpha \partial x^\beta} \right| &= C(\alpha, n_l)C(\beta, l)\left|x^{n_l-\beta}\xi^{l-\alpha}\right| \\ &= C\left|x^{n_l-\beta}\xi^{l-r-\alpha}\right| \left|\xi^r\right| \\ &= C\left|x^{n_l}\xi^{l-r}\right|^{1-\beta/n_l}\left|\xi^{-\alpha+\beta(l-r)/n_l}\right| \left|\xi\right|^r \\ &\leqq C\left|p(x,\xi)\xi^{-r}\right|^{1-\beta/n_l}\left|\xi\right|^r\left|\xi\right|^{-\alpha+\beta(l-r)/n_l}. \end{aligned} \tag{7.8}$$

But $|p(x,\xi)\xi^{-r}|$ is bounded away from zero, by (7.6). Hence we can replace $1-\beta/n_l$ by 1 in the exponent at the right hand side of (7.8). According to (2.2) and Lemma 6.1 bis, $n_l - l > n_r - r = -r$ for $l > r$, or $(l-r)/n_l < 1$. Setting $\delta = \max_{r<l\leqq m}(l-r)/n_l$, we find out that $\delta < 1$ and the estimate

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial\xi^\alpha} x^{n_l}\xi^l \right| \leqq C\left|p(x,\xi)\right| \left|\xi\right|^{-\alpha\rho+\delta\beta}$$

holds with $\rho = 1$. For $l \leqq r$ it follows immediately from (7.6) that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial\xi^\alpha} x^{n_l}\xi^l \right| \leqq C\left|p(x,\xi)\right| \left|\xi\right|^{-\alpha}$$

This finishes the proof of Theorem 3.

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